

GROUPS WITH INFINITELY MANY ENDS ARE NOT FRACTION GROUPS

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ABSTRACT. We show that any finitely generated group F with infinitely many ends is not a group of fractions of any finitely generated proper subsemigroup P , that is F cannot be expressed as a product PP^{-1} . In particular this solves a conjecture of Navas in the positive. As a corollary we obtain a new proof of the fact that finitely generated free groups do not admit isolated left-invariant orderings.

1 Introduction

The existence of a left-invariant order on a group G is equivalent to the existence of a positive cone $P \subset G$, that is a subsemigroup such that G can be written as a disjoint union $G = \{1\} \sqcup P \sqcup P^{-1}$. In fact there is a one-to-one correspondence between left-invariant orderings and such positive cones.

In this note we prove that whenever a finitely generated group F with infinitely many ends can be written as $F = PP^{-1}$, where P is a finitely generated subsemigroup of F , then $P = F$. Our result answers a question of Navas, who conjectured that finitely generated free groups are not groups of fractions of finitely generated subsemigroups P with $P \cap P^{-1} = \emptyset$.

As an application we obtain a new proof of the fact that the space of left-invariant orderings of a finitely generated free group (endowed with the Chabauty topology) does not have isolated points. This result follows from the work of McCleary [2], but appears in this form for the first time in the work of Navas [3]. It is worth noting that our proof is the first geometric one.

Our theorem complements a folklore result stating that whenever \mathcal{S} is a finite generating set for a group G , and G does not contain a free subsemigroup, then G is a group of fractions of P , the semigroup generated by \mathcal{S} .

We should note here that finitely generated groups with infinitely many ends have been classified by Stallings [5, 6]. They are precisely those fundamental groups of non-trivial graphs of groups with exactly one edge and a finite edge group, which are finitely generated and not virtually cyclic.

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2 The result

In the following we will use X to denote the Cayley graph of a finitely generated group F with respect to some finite generating set. We will identify F with vertices of X , and use d to denote the standard metric on the Cayley graph X .

We will assume that F has infinitely many ends, and so there will exist a constant K such that the ball $B = B(1, K)$ disconnects X into at least 3 components. We will use S to denote the set of vertices of B .

Definition 2.1. We say that $A \subset X$ is a *shoot* if and only if there exists $w \in F$ such that A is a connected component of $X \setminus wB$. We say that wB *bounds* the shoot.

Lemma 2.2. *Let $z \in F \setminus B$ and let A be a shoot bounded by B not containing z . Then there exists $w \in F$ such that wz and wB lie in A and wA contains B .*

Proof. Let $M = \{x \in F \mid d(xB, 1) \leq K\}$, and note that it is finite. Since F has infinitely many ends, there exists L such that

$$\Lambda = \{x \in F \mid d(1, x) = L\} \cap A$$

has at least $|M| + 2$ elements. Take $l \in \Lambda$. Then there exists $l' \in \Lambda$ such that $l'l^{-1} \notin M$.

Let $w = l'l^{-1}$. Observe that $w \notin M$ implies that $wB \cap B = \emptyset$. Also, $d(w, l') = d(1, l) = L$, and so $wB \subset A$.

Since $l \in A$, we see that $l' \in l'l^{-1}A = wA$. Suppose that $1 \notin wA$. Take a path between 1 and l' of length L . Since $1 \notin wA$, the path has to cross wB . But then

$$L \geq d(wB, l') + d(wB, 1) \geq L - K + K + 1 = L + 1,$$

which is a contradiction. We conclude that $1 \in wA$. Again, since $w \notin M$, we have $B \subset wA$.

As $z \notin A$, we have $wz \notin wA$. But $z \in F \setminus B$ and so wz lies in some shoot bounded by wB . Since $B \subset wA$ and $wB \subset A$, any shoot bounded by wB other than wA is contained in A . So $wz \in A$ and our proof is finished. \square

Theorem 2.3. *Let P be a finitely generated subsemigroup of a finitely generated group F with infinitely many ends. If $PP^{-1} = F$ then $P = F$.*

Proof. For ease of notation we will refer to the elements of P as positive, and to the elements of P^{-1} as negative.

We first note that any finite generating set of P is a generating set for F . Let X be the Cayley graph of F with respect to some such generating set.

We will use the notation K , B and S as defined above.

Step 1: We claim that $S(P^{-1} \cup \{1\}) = F$.

If P intersects each ball $B(x, K) = xB$ our claim is clear (after taking an inverse). So let us suppose that there exists an $x \in F$ such that

$$P \cap xB = \emptyset.$$

Let A_0 denote a shoot bounded by xB such that $1 \notin A_0$.

Let $z \in F \setminus S$ be any element, and let A be the shoot bounded by B containing z . We claim that we can find $y \in F$ such that $yz \in A_0$, $yB \subset A_0$, and yA does not contain B (see Figure 2.1).

Take any $y_0 \in F$ such that the first two properties hold. Then either the third one holds as well (in which case we take $y = y_0$), or we apply Lemma 2.2 to the triple z, B, A' , where A' is a shoot bounded by B other than A . We obtain an element w , such that

- $w'z \in y_0A' \subseteq A_0$;
- $w'B \subset y_0A' \subseteq A_0$; and
- $w'A'$ contains y_0B ,

where $w' = y_0w$. The last property tells us that $w'A$ does not contain B , and so $y = w'$ is as claimed.

Now, since $yz \in F = PP^{-1}$, we can write $yz = pq$, where p is positive and q is negative. Since there are no positive elements in xB by assumption, we see that $p \notin A_0$. But $yz \in A_0$, and therefore there is a negative word (a terminal subword of q) connecting a point in $B(x, K)$ to yz . This negative word (seen as a path) has to cross yB , since $yz \in yA$ which is separated from xB by yB . So there is a negative path from some vertex of yB to yz , and hence from an element of S to z (after translating by y^{-1}). We have thus shown that $z \in SP^{-1}$, and so

$$F \setminus S \subseteq SP^{-1}.$$

But clearly $S \subset S(P^{-1} \cup \{1\})$, and so we have proven the claim of step 1.

Step 2: We claim that $P = F$.

We have established above that $S(P^{-1} \cup \{1\}) = F$. Let Q be a minimal (with respect to cardinality) finite subset of F such that $Q(P^{-1} \cup \{1\}) = F$. Suppose that there exist distinct $q, q' \in Q$. Then $q^{-1}q' \in F = PP^{-1}$, and so $q^{-1}q' = ab^{-1}$ with $a, b \in P$. Hence

$$q, q' \in qaP^{-1},$$

and therefore we could replace Q by $(Q \cup \{qa\}) \setminus \{q, q'\}$ of smaller cardinality. This shows that $|Q| = 1$. Without loss of generality we can take $Q = \{1\}$, and thence get

$$P^{-1} \cup \{1\} = F.$$

Now let $f \in F \setminus \{1\}$. We have $f, f^{-1} \in P^{-1}$, and since P^{-1} is a semigroup, also $1 = ff^{-1} \in P^{-1}$. So $P^{-1} = F$. Taking an inverse concludes the theorem. \square

We now easily deduce the following.

Corollary 2.4. *Let F be a finitely generated group with infinitely many ends. Then F does not allow a left-invariant ordering with a finitely generated positive cone.*

Proof. Let P be the positive cone of a left-invariant ordering of F . Then $F = P \cup P^{-1} \cup \{1\}$, and so in particular $F = PP^{-1}$. But also $P \cap P^{-1} = \emptyset$, and so $P \neq F$. Now the contrapositive of Theorem 2.3 tells us that P is not finitely generated. \square

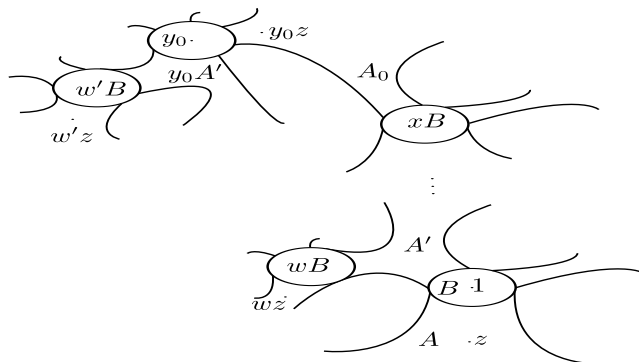


Figure 2.1: Step 1 of the theorem.

In fact the statement above follows from the work of Rivas [4], since left-orderable groups are torsion free.

We also get the following corollary.

Corollary 2.5. *The space of left-invariant orderings on any finitely generated free group has no isolated points.*

Proof. Let P be the positive cone of an isolated ordering of F , a finitely generated free group. By above, P is not finitely generated.

The order defined by P is isolated, and so there exists a finite set $S \subset F$ such that whenever we have another positive cone of an ordering P' such that $P \cap S = P' \cap S$, then $P = P'$. However the work of Smith and Clay [1, Theorem E] allows us to construct an order (in fact infinitely many such orders) whose positive cone P' satisfies $P \cap S = P' \cap S$, but such that $P \neq P'$. This is a contradiction. \square

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